Bundle gerbe module の splitting principle について

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1. Introduction

· Bundle gerbe(Murray):
a geometric realization of $H^3(X;\mathbb{Z})$

We can regard this as a higher generalization of a line bundle.

· Bundle gerbe module for a bundle gerbe
introduced by Bouwknegt-Carey-Mathai-Murray-Stevenson

regarded as a higher generalization of a complex vector bundle.

A complex vector bundle E
$\rightsquigarrow c(E)$: the Chern class
ch(E) : the Chern character

We can construct the twisted Chern character of bundle gerbe modules.

· classifying space and universal bundle
· splitting principle
· Chern-Weil construction ($\longleftarrow$ done by BCMMS)

Aim We would like to give an description of twisted Chern class/character in terms of Algebraic Topology.

This is an analogy of splitting principle.

2. Bundle gerbes
2.1 Bundle gerbes

Let

X : a smooth manifold
$Y \xrightarrow{\pi} X$: a fiber bundle over X
$Y^{[k]} := \{(y_1, \ldots, y_k) \in Y^k \mid \pi(y_1) = \cdots = \pi(y_k)\}$
$\pi_i : Y^{[k]} \to Y^{[k-1]}$
: the map which omits the i-th element
$L \to Y^{[2]}$: a hermitian line bundle

Definition.
(Y, L) : a bundle gerbe over X
$\Longleftrightarrow$
L is equipped with a product:

$$L_{(y_1,y_2)} \otimes L_{(y_2,y_3)} \xrightarrow{\cong} L_{(y_1,y_3)}$$

for $\forall (y_1, y_2), (y_2, y_3) \in Y^{[2]}$ which has associativity.

To fix a product of L is equivalent to specifying an isomorphism:

$$\pi_3^* L \otimes \pi_1^* L \xrightarrow{\cong} \pi_2^* L.$$

<u>Example</u>(Spin bundle gerbe).

Let

$Y \to X$: a $SO(n)$-bundle

$1 \to \mathbb{Z}_2 \to Spin(n) \xrightarrow{p} SO(n) \to 1$

: a central extension

Then we have a $\mathbb{Z}_2$-bundle over $Y^{[2]}$

$$Q := \{((y_1, y_2), \alpha) \in Y^{[2]} \times Spin(n) | p(\alpha) y_1 = y_2\}.$$

Let

L : the line bundle over $Y^{[2]}$

associated with $\mathbb{Z}_2$-bundle Q.

Then

(Y, L) is a bundle gerbe over X

called <u>the spin bundle gerbe</u> of Y.

Definition(trivialization).
$\eta \to Y$: a trivialization of (Y, L)
$\iff$
$\eta \to Y$: a hermitian line bundle and
$\pi_1^* \eta^* \otimes \pi_2^* \eta \cong L$.

A bundle gerbe (Y, L) is called trivial iff there is a trivialization η.

Definition(stable isomorphism).
Let $(Y, L)$and (Z, M) be bundle gerbes over X.
(Y, L) is stable isomorphic to (Z, M)
$\iff$
$(Y \times_\pi Z, L^* \otimes M)$: trivial

For every bundle gerbe (Y, L) over X, we have the Dixmier-Douady class $d(Y, L) \in H^3(X; \mathbb{Z})$.

Theorem(Murray).

$$d : \frac{\{\text{bundle gerbes over } X\}}{\text{stable iso.}} \xrightarrow{\cong} H^3(X; \mathbb{Z}),$$

Definition(bundle gerbe connection).

A hermitian connection ∇ on Y

is a bundle gerbe connection on (Y, L)

$\iff$

The product of L:

$$\pi_3^* L \otimes \pi_1^* L \xrightarrow{\cong} \pi_2^* L$$

preserves the connections:

$$\begin{array}{ccc} \pi_3^*\nabla \otimes 1 + 1 \otimes \pi_1^*\nabla & \text{on} & \pi_3^* L \otimes \pi_1^* L, \\ \pi_2^*\nabla & \text{on} & \pi_2^* L \end{array}$$

Definition.
We have the sequence of fiber products:

$$X \xleftarrow{\pi} Y \leftarrow Y^{[2]} \leftarrow Y^{[3]} \leftarrow \dots$$

and

$$0 \to \Omega^*(X) \xrightarrow{\delta} \Omega^*(Y) \xrightarrow{\delta} \Omega^*(Y^{[2]}) \to \cdots ,$$

where $\delta : \Omega^*(Y^{[k]}) \to \Omega^*(Y^{[k+1]})$ is defined by

$$\delta\omega = \sum_{i=1}^{k+1} (-1)^i \pi_i^* \omega.$$

Proposition.
This sequence is exact.

Remark. We have

$$dF(\nabla) = 0 \text{ and } \delta F(\nabla) = 0$$

and hence there is $f \in i\Omega^2(Y)$ satisfying

$$-\pi_1^* f + \pi_2^* f = F(\nabla).$$

We call such f a curving of (Y, L).

2.2 Bundle gerbe modules and the twisted Chern character

Let
(Y, L): a bundle gerbe over X
with a given bundle gerbe connection ∇
and curving f

$W \to Y$: a hermitian vector bundle

Definition(bundle gerbe module).
W : a bundle gerbe module for (Y, L)
$\Longleftrightarrow$
W is endowed with a multiplication of L:

$$L_{(y_1,y_2)} \otimes W_{y_2} \xrightarrow{\cong} W_{y_1}$$

for $\forall (y_1, y_2) \in Y^{[2]}$ which has the commutativity.

We denote by Mod(Y, L) the isomorphism classes of bundle gerbe modules for (Y, L).

To fix multiplication of L is equivalent to specifying an isomorphism

$$\varphi : L \otimes \pi_1^* W \to \pi_2^* W.$$

In the case that (Y, L) is trivial,we have

$$\pi_1^*\eta^* \otimes \pi_2^*\eta \otimes \pi_1^*W \cong L \otimes \pi_1^*W \cong \pi_2^*W$$
$$\rightsquigarrow \pi_1^*(W \otimes \eta^*) \cong \pi_2^*(W \otimes \eta^*)$$

So, a trivialization η of (Y, L) induces

$$/\eta : \mathrm{Mod}(Y, L) \to \mathrm{Vect}(X)$$

satisfying $\pi^*(W/\eta) = W \otimes \eta^*$.

Assumption

· the Dixmier-Douady class $d(Y, L)$ of (Y, L) is a torsion element.

i.e. $nd(Y, L) = 0$ for some n.

· the curving f is closed, i.e. $df = 0$.

Definition(bundle gerbe module connection).
A hermitian connection ∇^W on W is a bundle gerbe module connection for ∇

$\Longleftrightarrow$

The multiplication of L:

$$\varphi : L \otimes \pi_1^* W \to \pi_2^* W$$

preserves the connections:

$$\begin{array}{ccc} \nabla \otimes 1 + \pi_1^* \nabla^W & \text{on} & L \otimes \pi_1^* W, \\ \pi_2^* \nabla^W & \text{on} & \pi_2^* W \end{array}$$

Remark.
For every bundle gerbe module connection ∇^W,

$$F(\nabla) \otimes 1 + \pi_1^* F(\nabla^W) = \varphi \circ \pi_2^* F(\nabla^W) \circ \varphi^{-1}$$

which implies

$$\pi_1^*(f + F(\nabla^W) = \varphi \circ \pi_2^*(f + F(\nabla^W)) \circ \varphi^{-1}.$$

Therefore,

$$d \text{ tr}\left(\left(\frac{-1}{2\pi i}(f + F(\nabla^W))\right)^k\right) = 0$$

and

$$\pi_1^* \text{tr}\left(\left(\frac{-1}{2\pi i}(f + F(\nabla^W))\right)^k\right) = \pi_2^* \text{tr}\left(\left(\frac{-1}{2\pi i}(f + F(\nabla^W))\right)^k\right).$$

So, we have $\exists!$ closed form $\eta_k \in \Omega^{2k}(X)$ satisfying

$$\pi^* \eta_k = \text{tr}\left(\left(\frac{-1}{2\pi i}(f + F(\nabla^W))\right)^k\right).$$

for $\forall k$.

Definition(the twisted Chern character).
We define
the twisted Chern character $\mathrm{ch}^{\tau}_{\mathrm{DG}}(W)$ of W by

$$\mathrm{ch}^{\tau}_{\mathrm{DG}}(W) := \mathrm{rank}\ W + \sum_{k=1}^{\infty} \frac{1}{k!}[\eta_k] \in H^{2*}(X;\mathbb{R}).$$

Remark. The twisted Chern character $\mathrm{ch}^{\tau}_{\mathrm{DG}}$ is independent of the bundle gerbe connection and the bundle gerbe module connection but depends on the curving.

3. Splitting principle for bundle gerbe modules

3.1. Construction of splittings.

Definition(n-trivialization).
(Y, L) : a bundle gerbe over X
with $nd(Y, L) = 0$ for some n.
(hence, $(Y, L^{\otimes n})$: a trivial bundle gerbe)

η : an n-trivialization of (Y, L)
$\Longleftrightarrow$
η : a trivialization of $(Y, L^{\otimes n})$

Remark.
(Y, L) : a bundle gerbe over X
given an n-trivialization η
Then η induces a semi-group homomorphism

$$\mathrm{Mod}(Y, L) \xrightarrow{\otimes n} \mathrm{Mod}(Y, L^{\otimes n}) \xrightarrow{/\eta} \mathrm{Vect}(X)$$

which satisfies

$$\pi^*(W^{\otimes n}/\eta) = W^{\otimes n} \otimes \eta^*.$$

Take the projectivization

$$\widetilde{\mathbb{P}}(W) := \coprod_{y \in Y} \{y\} \times \mathbb{P}(W_y)$$

of a bundle gerbe module W for (Y, L).

<u>Remark</u>.
The multiplication of L for W

$$L_{(y_1,y_2)} \otimes W_{y_2} \xrightarrow{\cong} W_{y_1}$$

induces isomorphisms

$$\varphi_{(y_1,y_2)} : \mathbb{P}(W_{y_2}) \to \mathbb{P}(W_{y_1})$$

<u>Definition</u>
We define $\mathbb{P}(W)$ by

$$\mathbb{P}(W) := \widetilde{\mathbb{P}}(W)/\sim$$

Here, we define

$$(y_1, [w_1]) \sim (y_2, [w_2]) \Longleftrightarrow \varphi_{(y_1,y_2)}([w_2]) = [w_1].$$

Remark.

We have a commutative diagram:

$$\begin{array}{ccc} Y & \xleftarrow{\;\overline{p}\;} & \widetilde{\mathbb{P}}(W) \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\ X & \xleftarrow{\;p\;} & \mathbb{P}(W) \end{array}$$

Definition.

We define a bundle gerbe $(\widetilde{\mathbb{P}}(W), \widetilde{L})$ over $\mathbb{P}(W)$ by

$$\widetilde{L} := \overline{p}^*(L).$$

That is,

$$\begin{array}{ccc} & & \widetilde{L} \\ & & \Big\downarrow \\ \widetilde{\mathbb{P}}(W) & \longleftarrow & \widetilde{\mathbb{P}}(W)^{[2]} \\ \Big\downarrow{\scriptstyle \pi} & & \\ \mathbb{P}(W) & & . \end{array}$$

Proposition.
Let γ_W be the tautological line bundle over $\widetilde{\mathbb{P}}(W)$ defined by

$$\gamma_W = \{(y, l, w) \in \overline{p}^*(W) \mid y \in Y, l \in \mathbb{P}(W_y), w \in l\}.$$

Then, γ_W and $W^\perp$ are bundle gerbe modules for $(\widetilde{\mathbb{P}}(W), \widetilde{L})$.

Hence, we obtain the splitting of a bundle gerbe module W:

$$\overline{p}^*W = \gamma_W \oplus W^\perp.$$

We call γ_W the tautological bundle gerbe module of W.

3.2. the twisted Chern classes.

<u>**Definition**</u>(the twisted Euler class)
(Y, L): a bundle gerbe over X
given an n-trivialization η
ξ: a bundle gerbe module for (Y, L)
with rank$\xi = 1$.
We define <u>the twisted Euler class</u> $\chi^\tau(\xi)$ by

$$\chi^\tau(\xi) := \frac{1}{n}((\xi^{\otimes n})/\eta).$$

By using this, we can define a homomorphism θ of degree 0:

$$\theta : H^*(\mathbb{CP}^{m-1}; \mathbb{Q}) \to H^*(\mathbb{P}(W); \mathbb{Q})$$

by $\theta(\chi(\gamma(\mathbb{CP}^{m-1}))^k) = \chi^\tau(\gamma_W)^k$ for $\forall k$.

Consider

$$i_x : \mathbb{P}(W)_x \hookrightarrow \mathbb{P}(W).$$

It is easy to see that for every $x \in X$,

$$i_x^* \circ \theta : H^*(\mathbb{CP}^{m-1}; \mathbb{Q}) \to H^*(\mathbb{P}(W)_x; \mathbb{Q})$$

is isomorphism.

Therefore, by using the Leray-Hirsch theorem we obtain the following:

Theorem. We have the isomorphism of graded modules

$$\Phi : H^*(X;\mathbb{Q}) \otimes H^*(\mathbb{CP}^{m-1};\mathbb{Q}) \to H^*(\mathbb{P}(W);\mathbb{Q})$$

defined by $\Phi(\beta \otimes \alpha) = p^*\beta \cup \theta(\alpha)$.

Hence, there is an unique m-tuple

$$(\beta_1, \ldots, \beta_m) \in \prod_{i=k}^{m} H^{2k}(X;\mathbb{Q})$$

satisfying

$$-\chi^\tau(\gamma_W)^m = \sum_{k=1}^{m} (-1) p^*\beta_k \cup \chi^\tau(\gamma_W)^{m-k}$$

Definition We define the twisted Chern class of W by

$$c_k^\tau(W) = \begin{cases} 1 & \text{if } k = 0, \\ \beta_k & \text{if } 1 \le k \le m, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition.

1. (naturality)

$$c^\tau(\overline{f}^* W) = f^* c^\tau(W)$$

for every smooth map $f : X \to Z$ and bundle gerbe module W.

2. $\exists p : \hat{X} \to X$ such that

$$p^* : H^*(X; \mathbb{Q}) \to H^*(\hat{X}; \mathbb{Q}) : \text{ injective}$$

and $\overline{p}^* W$ splits into m bundle gerbe modules ξ_i of rank 1:

$$\overline{p}^* W \cong \xi_1 \oplus \cdots \oplus \xi_m$$

3. $c^\tau(V \oplus W) = c^\tau(V) \cup c^\tau(W)$.

Let

σ_k : the k-th elementary symmetric polynomial in m variables

s_k: the k-th Newton polynomial in m variables.

These satisfy

$$t_1^k + \cdots + t_m^k = s_k(\sigma_1, \ldots, \sigma_k).$$

<u>**Definition**</u>(the twisted Chern character in algebraic topology).

We define <u>the twisted Chern character</u> $\mathrm{ch}^{\tau}_{\mathrm{AT}}(W)$ in terms of Algebraic Topology by

$$\mathrm{ch}^{\tau}_{\mathrm{AT}}(W) := \mathrm{rank}\ W + \sum_k \frac{1}{k!} s_k(c^{\tau}_1(W), \ldots, c^{\tau}_k(W)).$$

<u>**Properties**</u>.

1. $\mathrm{ch}^{\tau}_{\mathrm{AT}}(V \oplus W) = \mathrm{ch}^{\tau}_{\mathrm{AT}}(V) + \mathrm{ch}^{\tau}_{\mathrm{AT}}(W)$.
2. $\mathrm{ch}^{\tau}_{\mathrm{AT}}(V \otimes W) = \mathrm{ch}^{\tau}_{\mathrm{AT}}(V) \cup \mathrm{ch}^{\tau}_{\mathrm{AT}}(W)$.
3. $\mathrm{ch}(W^{\otimes n}/\eta) = \mathrm{ch}^{\tau}_{\mathrm{AT}}(W)^n$.

Definition(compatible curving).
(Y, L) : a bundle gerbe over X
with a given n-trivialization η
A curving f of (Y, L) is compatible with the n-trivialization η
$\iff$
$\exists\nabla^\eta$: a hermitian connection on η which satisfies

$$f = F(\nabla^\eta)/n.$$

Theorem(T).
(Y, L) : a bundle gerbe over X
with a given n-trivialization η
and a compatible curving f with η.

Then for every bundle gerbe module W, we have

$$\mathrm{ch}^{\tau}_{\mathrm{DG}}(W) = \mathrm{ch}^{\tau}_{\mathrm{AT}}(W).$$